Correlation-based imaging in random media

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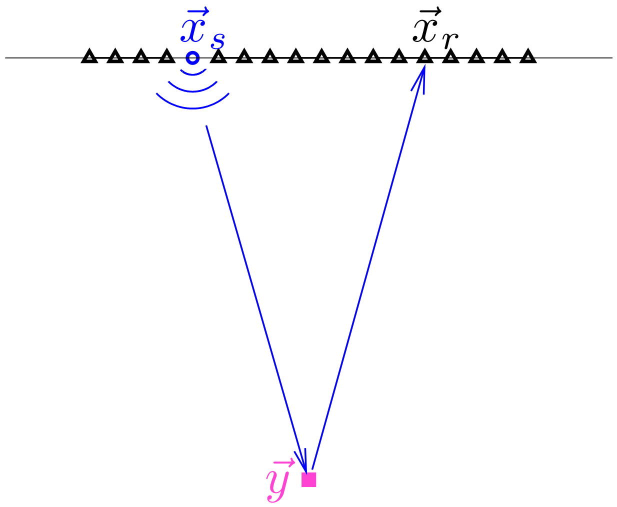
with George Papanicolaou (Stanford University), Knut Sølna (UC Irvine).

- Principle of sensor array imaging:

- probe an unknown medium with waves,
- record the waves transmitted through or reflected by the medium,
- process the recorded data to extract relevant information about some features of the medium.

Reflector imaging through a homogeneous medium



• Sensor array imaging of a reflector located at $\vec{\boldsymbol{y}}$. $\vec{\boldsymbol{x}}_s$ is a source, $\vec{\boldsymbol{x}}_r$ is a receiver. Measured data: $\{u(t, \vec{\boldsymbol{x}}_r; \vec{\boldsymbol{x}}_s), r = 1, \ldots, N_\mathrm{r}, s = 1, \ldots, N_\mathrm{s}\}$.

• Mathematical model:

$$\left(\frac{1}{c_0^2} + \frac{1}{c_{\mathrm{ref}}^2} \mathbf{1}_{B_{\mathrm{ref}}}(\vec{\boldsymbol{x}} - \vec{\boldsymbol{y}})\right) \frac{\partial^2 u}{\partial t^2}(t, \vec{\boldsymbol{x}}; \vec{\boldsymbol{x}}_\mathrm{s}) - \Delta_{\vec{\boldsymbol{x}}} u(t, \vec{\boldsymbol{x}}; \vec{\boldsymbol{x}}_\mathrm{s}) = f(t)\delta(\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}_\mathrm{s})$$

• Purpose of imaging: using the measured data, build an imaging function $\mathcal{I}(\vec{\boldsymbol{y}}^S)$ that would ideally look like $\frac{1}{c_{\mathrm{ref}}^2} \mathbf{1}_{B_{\mathrm{ref}}}(\vec{\boldsymbol{y}}^S - \vec{\boldsymbol{y}})$, in order to extract the relevant information $(\vec{\boldsymbol{y}}, B_{\mathrm{ref}}, c_{\mathrm{ref}})$ about the reflector.

- Classical imaging functions:

1) Least-Squares imaging: minimize the quadratic misfit between measured data and synthetic data obtained by solving the wave equation with a candidate $(\vec{\boldsymbol{y}}_{\rm test}, B_{\rm test}, c_{\rm test})$.

2) Reverse Time imaging: simplify Least-Squares imaging by “linearization” of the forward problem.

3) Kirchhoff Migration: simplify Reverse Time imaging by substituting travel time migration for full wave equation.

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3) Kirchhoff Migration: simplify Reverse Time imaging by substituting travel time migration for full wave equation.

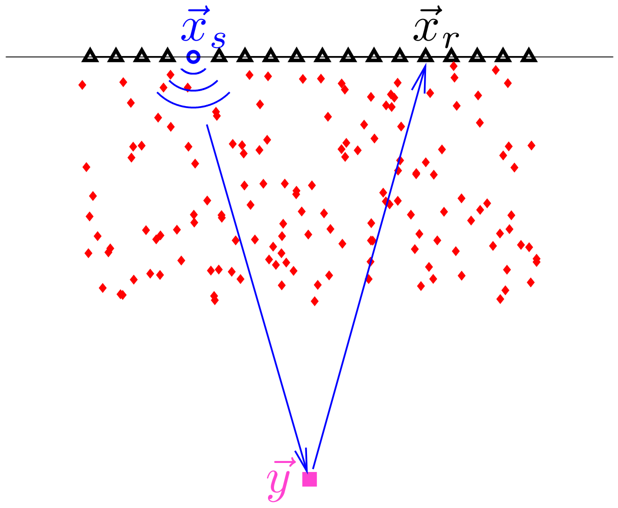
• Kirchhoff Migration function:

$$\mathcal{I}_{\text{KM}}(\vec{\boldsymbol{y}}^S) = \sum_{r=1}^{N_\text{r}} \sum_{s=1}^{N_\text{s}} u\big(\mathcal{T}(\vec{\boldsymbol{x}}_s, \vec{\boldsymbol{y}}^S) + \mathcal{T}(\vec{\boldsymbol{y}}^S, \vec{\boldsymbol{x}}_r), \vec{\boldsymbol{x}}_r; \vec{\boldsymbol{x}}_s\big)$$

It forms the image with the superposition of the backpropagated traces.
$\mathcal{T}(\vec{\boldsymbol{y}}^S, \vec{\boldsymbol{x}})$ is the travel time from $\vec{\boldsymbol{x}}$ to $\vec{\boldsymbol{y}}^S$, i.e. $\mathcal{T}(\vec{\boldsymbol{y}}^S, \vec{\boldsymbol{x}}) = |\vec{\boldsymbol{y}}^S - \vec{\boldsymbol{x}}|/c_0$.
- Very robust with respect to measurement noise [1].
- Sensitive to clutter noise (due to scattering medium): If the medium is scattering, then Kirchhoff Migration (usually) does not work.

[1] H. Ammari, J. Garnier, and K. Sølna, *Waves in Random and Complex Media* **22**, 40 (2012).

Reflector imaging through a scattering medium



- Sensor array imaging of a reflector located at $\vec{\boldsymbol{y}}$. $\vec{\boldsymbol{x}}_s$ is a source, $\vec{\boldsymbol{x}}_r$ is a receiver.

Data: $\{u(t, \vec{\boldsymbol{x}}_r; \vec{\boldsymbol{x}}_s), r = 1, \dots, N_{\rm r}, s = 1, \dots, N_{\rm s}\}$.

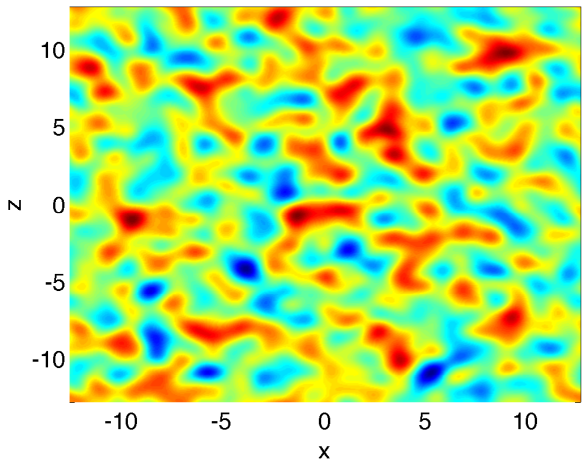
$$\left(\frac{1}{c^2(\vec{\boldsymbol{x}})} + \frac{1}{c_{\rm ref}^2}\mathbf{1}_{B_{\rm ref}}(\vec{\boldsymbol{x}} - \vec{\boldsymbol{y}})\right)\frac{\partial^2 u}{\partial t^2}(t, \vec{\boldsymbol{x}}; \vec{\boldsymbol{x}}_{\rm s}) - \Delta_{\vec{\boldsymbol{x}}} u(t, \vec{\boldsymbol{x}}; \vec{\boldsymbol{x}}_{\rm s}) = f(t)\delta(\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}_{\rm s})$$

- Random medium model:

$$\frac{1}{c^2(\vec{\boldsymbol{x}})} = \frac{1}{c_0^2}\big(1 + \mu(\vec{\boldsymbol{x}})\big)$$

c_0 is a reference speed,

$\mu(\vec{\boldsymbol{x}})$ is a zero-mean random process.



Imaging through a randomly scattering medium: strategy

• A multiscale analysis is possible in different asymptotic regimes (small wavelength, large propagation distance, small correlation length, ...).

• In the limit the wave equation with random coefficients is replaced by a stochastic partial differential equation driven by Brownian fields; for instance, an Itô-Schrödinger equation in the paraxial regime.

• Stochastic calculus can then be used.

• Compute the mean and variance of an imaging function $\mathcal{I}(\vec{\boldsymbol{y}}^S)$.

$\hookrightarrow$ resolution and stability analysis.

• The mean imaging function $\vec{\boldsymbol{y}}^S \to \mathbb{E}\big[\mathcal{I}(\vec{\boldsymbol{y}}^S)\big]$ characterizes the precision in the localization and characterization of the reflector (resolution).

• Criterium for statistical stability:

$$\mathrm{SNR} := \frac{\mathbb{E}\big[\mathcal{I}(\vec{\boldsymbol{y}}^S)\big]}{\mathrm{Var}\big(\mathcal{I}(\vec{\boldsymbol{y}}^S)\big)^{1/2}} > 1$$

$\hookrightarrow$ design the imaging function to get good trade-off between stability and resolution.

- General results obtained by a multiscale analysis.

- The mean wave is small while the wave fluctuations are large.

$\Longrightarrow$ The Kirchhoff Migration function (or Reverse Time imaging function) is unstable in randomly scattering media.

- The wave fluctuations at nearby points and nearby frequencies are correlated. The wave correlations carry information about the medium.

$\Longrightarrow$ One can use local cross correlations for imaging.

- More detailed results depend on the scattering regime.

Wave propagation in the random paraxial regime

- Consider the time-harmonic form of the scalar wave equation ($\vec{\boldsymbol{x}} = (\boldsymbol{x}, z)$)

$$(\partial_z^2 + \Delta_\perp)\hat{u} + \frac{\omega^2}{c_0^2}\big(1 + \mu(\boldsymbol{x}, z)\big)\hat{u} = 0.$$

Consider the paraxial regime "$\lambda \ll l_c \ll L$":

$$\omega \to \frac{\omega}{\varepsilon^4}, \qquad \mu(\boldsymbol{x}, z) \to \varepsilon^3 \mu\Big(\frac{\boldsymbol{x}}{\varepsilon^2}, \frac{z}{\varepsilon^2}\Big).$$

The function $\hat{\phi}^\varepsilon$ (slowly-varying envelope of a plane wave) defined by

$$\hat{u}^\varepsilon(\omega, \boldsymbol{x}, z) = e^{i\frac{\omega z}{\varepsilon^4 c_0}} \hat{\phi}^\varepsilon\Big(\omega, \frac{\boldsymbol{x}}{\varepsilon^2}, z\Big)$$

satisfies

$$\varepsilon^4 \partial_z^2 \hat{\phi}^\varepsilon + \left(2i\frac{\omega}{c_0}\partial_z \hat{\phi}^\varepsilon + \Delta_\perp \hat{\phi}^\varepsilon + \frac{\omega^2}{c_0^2}\frac{1}{\varepsilon}\mu\Big(\boldsymbol{x}, \frac{z}{\varepsilon^2}\Big)\hat{\phi}^\varepsilon\right) = 0.$$

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- In the regime $\varepsilon \ll 1$, the forward-scattering approximation in direction z is valid and $\hat{\phi} = \lim_{\varepsilon \to 0} \hat{\phi}^\varepsilon$ satisfies the Itô-Schrödinger equation [1]

$$2i\frac{\omega}{c_0}\partial_z \hat{\phi} + \Delta_\perp \hat{\phi} + \frac{\omega^2}{c_0^2}\dot{B}(\boldsymbol{x}, z)\hat{\phi} = 0$$

with $B(\boldsymbol{x}, z)$ Brownian field $\mathbb{E}[B(\boldsymbol{x}, z)B(\boldsymbol{x}', z')] = \gamma(\boldsymbol{x} - \boldsymbol{x}') \min(z, z')$,
$\gamma(\boldsymbol{x}) = \int_{-\infty}^{\infty} \mathbb{E}[\mu(\boldsymbol{0}, 0)\mu(\boldsymbol{x}, z)]dz$.

[1] J. Garnier and K. Sølna, *Ann. Appl. Probab.* **19**, 318 (2009).

Wave propagation in the random paraxial regime

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$$d\hat{\phi} = \frac{ic_0}{2\omega}\Delta_\perp \hat{\phi} dz + \frac{i\omega}{2c_0}\hat{\phi} \circ dB(\boldsymbol{x}, z)$$

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with $B(\boldsymbol{x}, z)$ Brownian field $\mathbb{E}[B(\boldsymbol{x}, z)B(\boldsymbol{x}', z')] = \gamma(\boldsymbol{x} - \boldsymbol{x}') \min(z, z')$,
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- We introduce the fundamental solution $\hat{G}\big(\omega,(\boldsymbol{x},z),(\boldsymbol{x}_0,z_0)\big)$:

$$d\hat{G} = \frac{ic_0}{2\omega}\Delta_\perp \hat{G}dz + \frac{i\omega}{2c_0}\hat{G}\circ dB(\boldsymbol{x},z)$$

starting from $\hat{G}\big(\omega,(\boldsymbol{x},z=z_0),(\boldsymbol{x}_0,z_0)\big) = \delta(\boldsymbol{x}-\boldsymbol{x}_0)$.

- In a homogeneous medium ($B \equiv 0$) the fundamental solution is

$$\hat{G}_0\big(\omega,(\boldsymbol{x},z),(\boldsymbol{x}_0,z_0)\big) = \frac{\exp\left(\frac{i\omega|\boldsymbol{x}-\boldsymbol{x}_0|^2}{2c_0|z-z_0|}\right)}{2i\pi c_0\frac{|z-z_0|}{\omega}}.$$

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- In a random medium, by Itô's formula

$$\mathbb{E}\big[\hat{G}\big(\omega, (\boldsymbol{x}, z), (\boldsymbol{x}_0, z_0)\big)\big] = \hat{G}_0\big(\omega, (\boldsymbol{x}, z), (\boldsymbol{x}_0, z_0)\big) \exp\Big(-\frac{\gamma(\mathbf{0})\omega^2|z - z_0|}{8c_0^2}\Big),$$

where $\gamma(\boldsymbol{x}) = \int_{-\infty}^{\infty} \mathbb{E}[\mu(\mathbf{0}, 0)\mu(\boldsymbol{x}, z)]dz$.

- Strong damping of the mean wave.

$\Longrightarrow$ Reverse Time imaging and Kirchhoff migration fail.

- In a random medium, by Itô's formula

$$\mathbb{E}\Big[\hat{G}\big(\omega,(\boldsymbol{x},z),(\boldsymbol{x}_0,z_0)\big)\overline{\hat{G}\big(\omega,(\boldsymbol{x}',z),(\boldsymbol{x}_0,z_0)\big)}\Big]$$
$$= \hat{G}_0\big(\omega,(\boldsymbol{x},z),(\boldsymbol{x}_0,z_0)\big)\overline{\hat{G}_0\big(\omega,(\boldsymbol{x}',z),(\boldsymbol{x}_0,z_0)\big)}\exp\Big(-\frac{\gamma_2(\boldsymbol{x}-\boldsymbol{x}')\omega^2|z-z_0|}{4c_0^2}\Big),$$

where $\gamma_2(\boldsymbol{x}) = \int_0^1 \gamma(\mathbf{0}) - \gamma(\boldsymbol{x}s)ds$ (note $\gamma_2(\mathbf{0}) = 0$).

- The fields at nearby points are correlated.
- Same results in frequency: The fields at nearby frequencies are correlated.

$\Longrightarrow$ One should migrate local cross correlations for imaging.

• In a random medium, by Itô's formula

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• Same results in frequency: The fields at nearby frequencies are correlated.

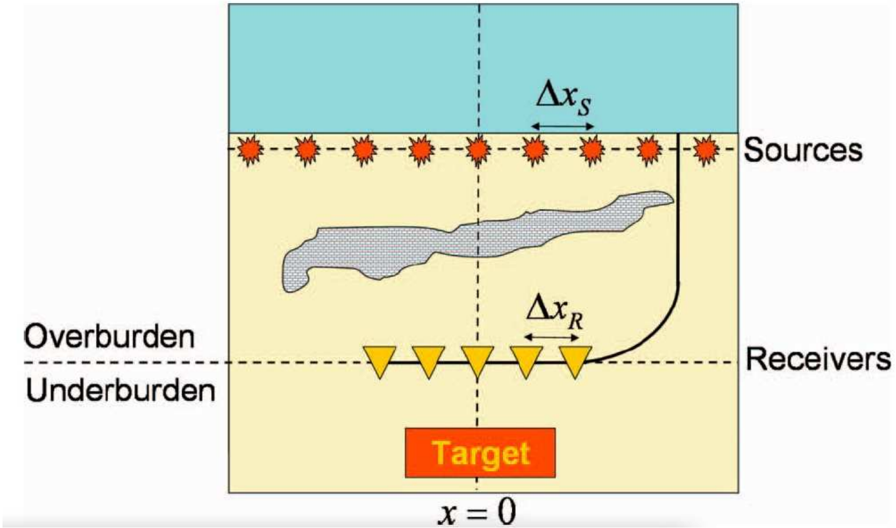
$\Longrightarrow$ One should migrate local cross correlations for imaging.

• In a random medium, by Itô's formula, one can write a closed-form equation for the n-th order moment.

Depending on the statistics of the random medium, the wave fluctuations may have Gaussian statistics or not [1].

[1] J. Garnier and K. Sølna, to appear in Comm. Part. Differ. Equat.

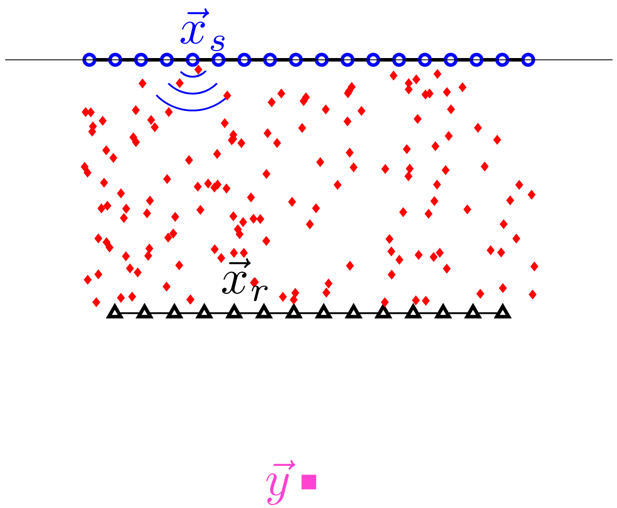
Application: Imaging below an "overburden"



Imaging below an "overburden"

From van der Neut and Bakulin (2009)

Imaging below an overburden

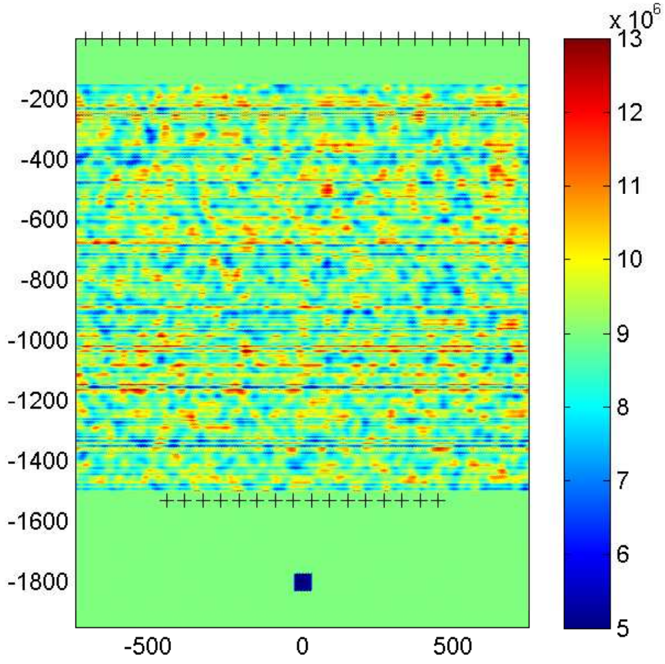


Array imaging of a reflector at $\vec{\boldsymbol{y}}$. $\vec{\boldsymbol{x}}_s$ is a source, $\vec{\boldsymbol{x}}_r$ is a receiver.
Data: $\{u(t, \vec{\boldsymbol{x}}_r; \vec{\boldsymbol{x}}_s), r = 1, \ldots, N_\mathrm{r}, s = 1, \ldots, N_\mathrm{s}\}$.

If the "overburden" is scattering, then Kirchhoff Migration does not work:

$$\mathcal{I}_{\mathrm{KM}}(\vec{\boldsymbol{y}}^S) = \sum_{r=1}^{N_\mathrm{r}} \sum_{s=1}^{N_\mathrm{s}} u\Big(\mathcal{T}(\vec{\boldsymbol{x}}_s, \vec{\boldsymbol{y}}^S) + \mathcal{T}(\vec{\boldsymbol{y}}^S, \vec{\boldsymbol{x}}_r), \vec{\boldsymbol{x}}_r; \vec{\boldsymbol{x}}_s\Big)$$

Numerical simulations



Computational setup

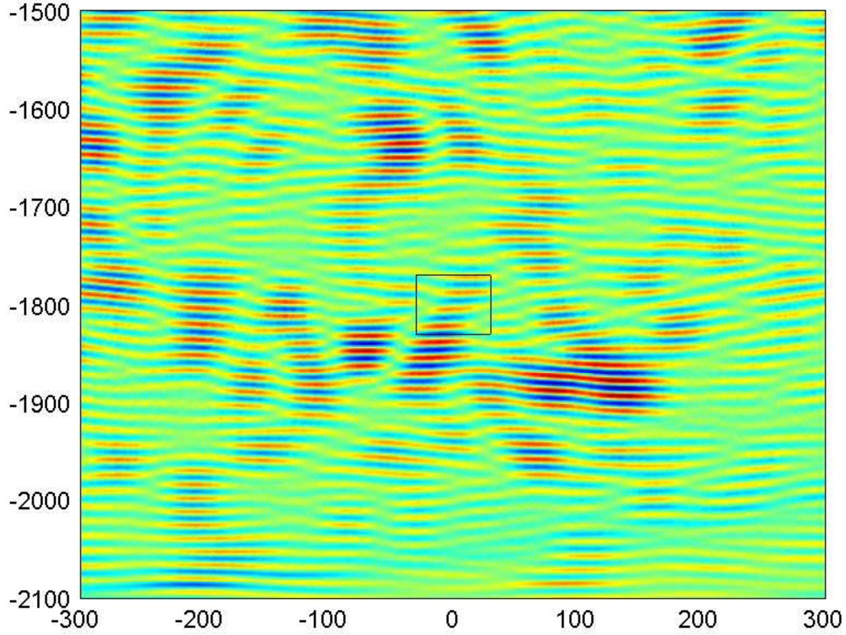
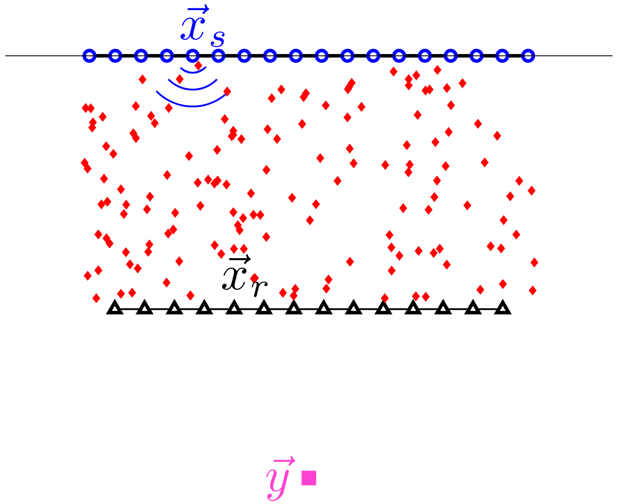


Image obtained with Kirchhoff Migration

(simulations carried out by Chrysoula Tsogka)

Imaging below an overburden



$\vec{\boldsymbol{x}}_s$ is a source, $\vec{\boldsymbol{x}}_r$ is a receiver. Data: $\{u(t, \vec{\boldsymbol{x}}_r; \vec{\boldsymbol{x}}_s), r = 1, \ldots, N_\mathrm{r}, s = 1, \ldots, N_\mathrm{s}\}$.

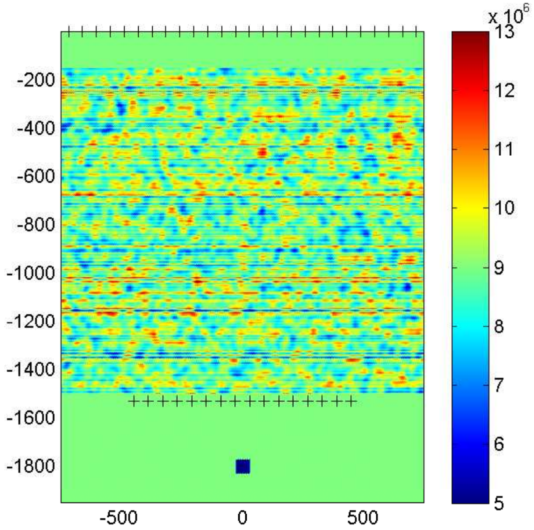
Image with Kirchhoff Migration of the cross correlation matrix:

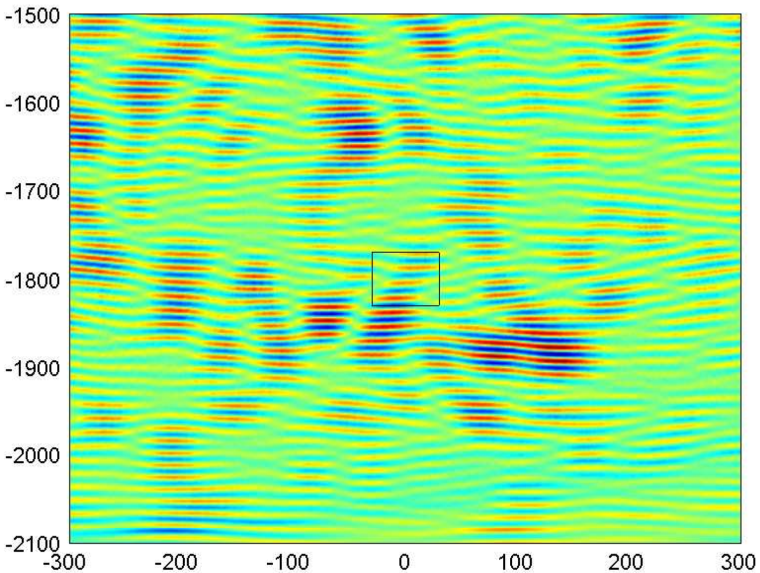
$$\mathcal{I}(\vec{\boldsymbol{y}}^S) = \sum_{r,r'=1}^{N_\mathrm{r}} \mathcal{C}\big(\mathcal{T}(\vec{\boldsymbol{x}}_r, \vec{\boldsymbol{y}}^S) + \mathcal{T}(\vec{\boldsymbol{y}}^S, \vec{\boldsymbol{x}}_{r'}), \vec{\boldsymbol{x}}_r, \vec{\boldsymbol{x}}_{r'}\big),$$

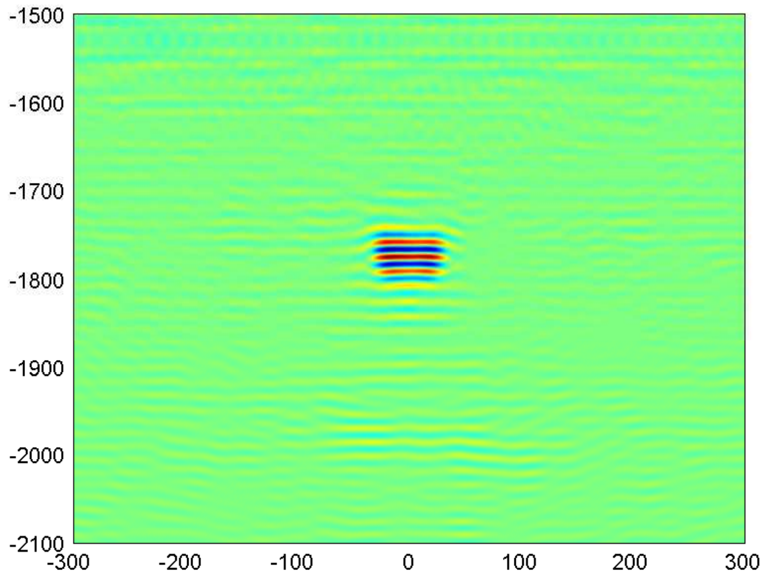
with

$$\mathcal{C}(\tau, \vec{\boldsymbol{x}}_r, \vec{\boldsymbol{x}}_{r'}) = \sum_{s=1}^{N_\mathrm{s}} \int u(t, \vec{\boldsymbol{x}}_r; \vec{\boldsymbol{x}}_s) u(t+\tau, \vec{\boldsymbol{x}}_{r'}; \vec{\boldsymbol{x}}_s) dt \ , \qquad r, r' = 1, \ldots, N_\mathrm{r}$$

Numerical simulations







Kirchhoff Migration

Cross Correlation Migration

Analysis in randomly scattering media

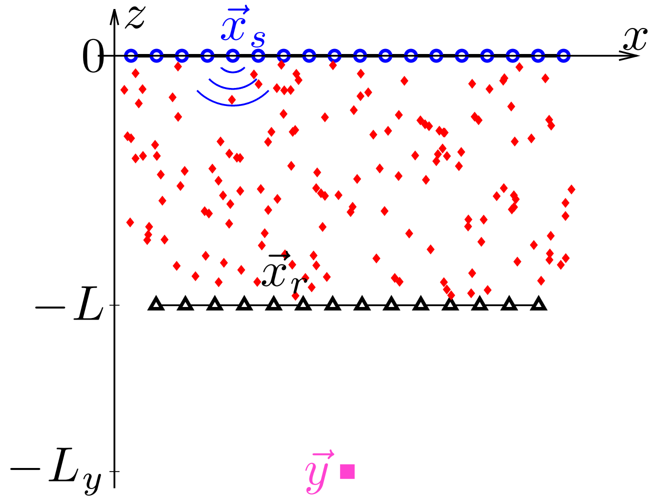
• Does the cross correlation imaging function give good images in scattering media ?
↪ It is possible to analyze the resolution and stability of the imaging function in randomly scattering media:
- analysis in the random paraxial regime,
- analysis in the randomly layered regime,
- analysis in the radiative transfer regime.

• General results:
Imaging function is stable provided the bandwidth is large enough and/or the source array is large enough.
Resolution is essentially independent of the size of the source array.

• Detailed results: Clarify the role of scattering.
- in the random paraxial regime, scattering helps (it enhances the angular diversity of the illumination).
- in the randomly layered regime, scattering does not help (it reduces the angular diversity of the illumination).

[1] J. Garnier and G. Papanicolaou, *Inverse Problems* **28** 075002 (2012).

Imaging below an overburden: analysis in the paraxial regime



- Assume that:

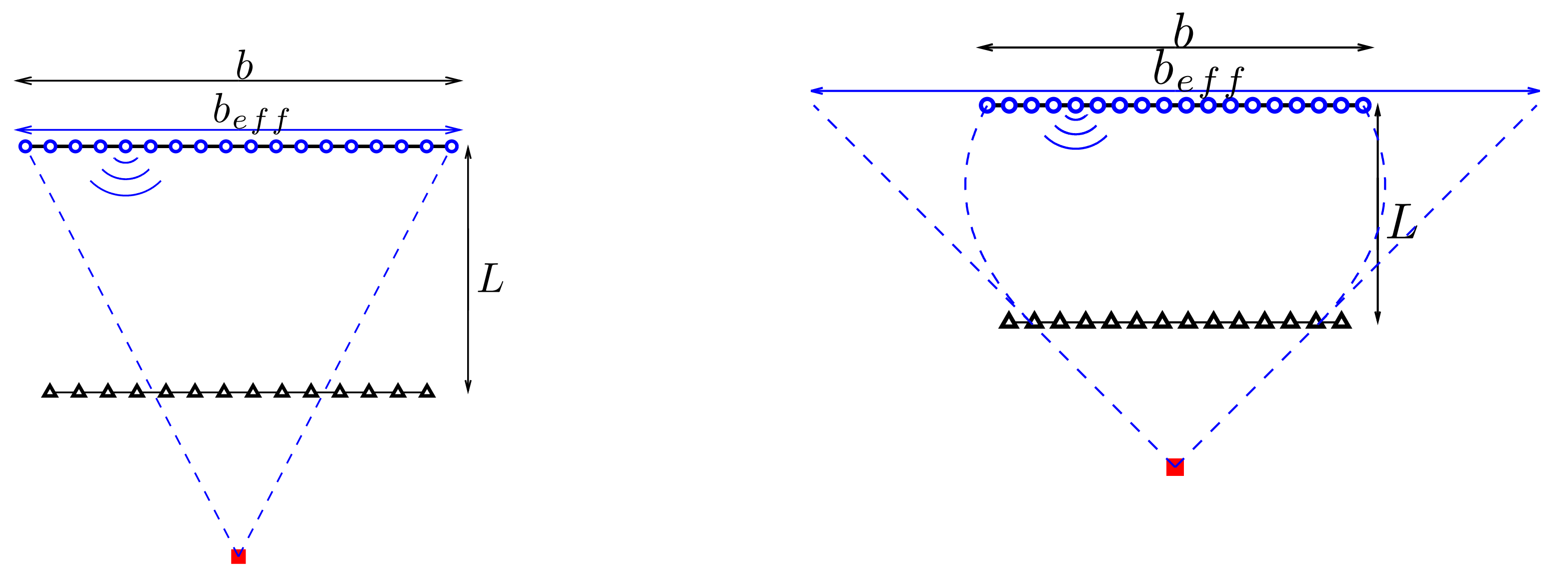
- the source aperture is b and the receiver aperture is a.

- there is a point reflector at $\vec{\boldsymbol{y}} = (\boldsymbol{y}, -L_y)$.

- the covariance function $\gamma(\boldsymbol{x}) = \int \mathbb{E}[\mu(\mathbf{0}, 0)\mu(\boldsymbol{x}, z)]dz$ can be expanded as $\gamma(\boldsymbol{x}) = \gamma(\mathbf{0}) - \bar{\gamma}_2|\boldsymbol{x}|^2 + o(|\boldsymbol{x}|^2)$ for small $|\boldsymbol{x}|$.

- scattering is strong: $\frac{\gamma(\mathbf{0})\omega_0^2 L}{c_0^2} > 1$ ($\rightarrow$ mean wave is damped).

Imaging below an overburden: analysis in the paraxial regime



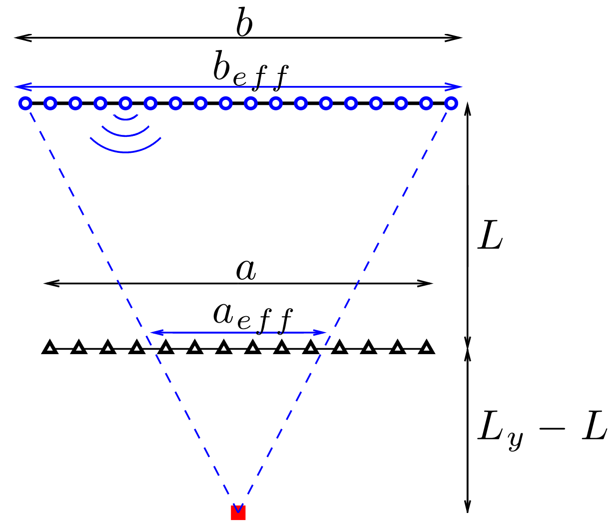
Homogeneous medium | Random medium

Effective source aperture:

$$b_{\text{eff}} = b$$

$$b_{\text{eff}} = \left(b^2 + \frac{\bar{\gamma}_2 L^3}{3} \right)^{1/2}$$

Imaging below an overburden: analysis in the paraxial regime



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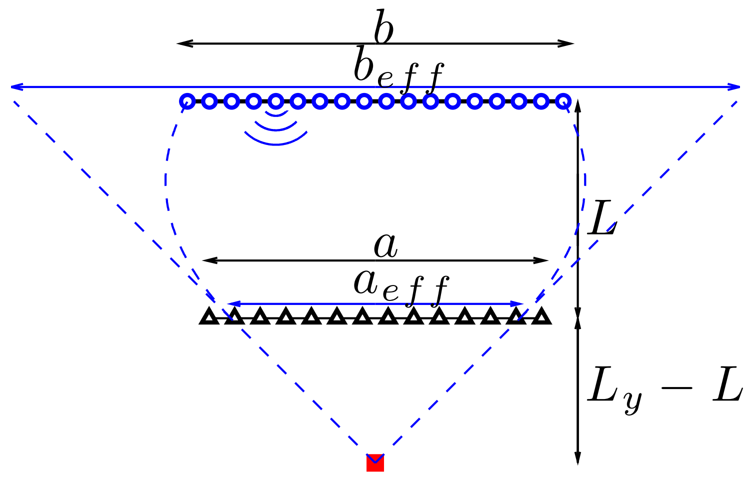


Random medium

Effective source aperture:

$$b_{\text{eff}} = b \qquad\qquad b_{\text{eff}} = \Big(b^2 + \frac{\bar{\gamma}_2 L^3}{3} \Big)^{1/2}$$

Effective receiver aperture:

$$a_{\text{eff}} = b \frac{L_y - L}{L_y} \qquad\qquad a_{\text{eff}} = b_{\text{eff}} \frac{L_y - L}{L_y}$$

Imaging below an overburden: analysis in the paraxial regime

• The imaging function for the search point $\vec{\boldsymbol{y}}^S$ is

$$\mathcal{I}(\vec{\boldsymbol{y}}^S) = \frac{1}{N_{\rm r}^2} \sum_{r,r'=1}^{N_{\rm r}} \mathcal{C}\big(\mathcal{T}(\vec{\boldsymbol{x}}_r, \vec{\boldsymbol{y}}^S) + \mathcal{T}(\vec{\boldsymbol{y}}^S, \vec{\boldsymbol{x}}_{r'}), \vec{\boldsymbol{x}}_r, \vec{\boldsymbol{x}}_{r'}\big)$$

• The imaging function is statistically stable ($\lambda_0 \ll b \ll L$).

• The lateral resolution is $\dfrac{\lambda_0(L_y - L)}{a_{\rm eff}}$. The range resolution is $\dfrac{c_0}{B}$.
Here: λ_0 is the carrier wavelength, B is the bandwidth.

• Since $a_{\rm eff}\ |_{\rm rand} > a_{\rm eff}\ |_{\rm homo}$, this shows that scattering helps.
- physical reason: scattering enhances the angular diversity of the illumination.
- effect already noticed for time-reversal experiments, in which the recorded waves are time-reversed and sent back in the real medium.

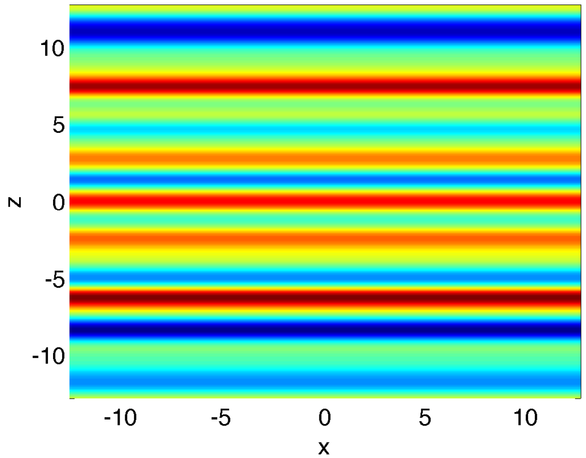
Randomly layered medium

- Random medium model ($\vec{\boldsymbol{x}} = (\boldsymbol{x}, z)$):

$$\frac{1}{c^2(\vec{\boldsymbol{x}})} = \frac{1}{c_0^2}\big(1 + \mu(z)\big)$$

c_0 is a reference speed,

$\mu(z)$ is a zero-mean random process.



- Consider the time-harmonic form of the scalar wave equation ($\vec{\boldsymbol{x}} = (\boldsymbol{x}, z)$)

$$(\partial_z^2 + \Delta_\perp)\hat{u} + \frac{\omega^2}{c_0^2}\big(1 + \mu(z)\big)\hat{u} = 0$$

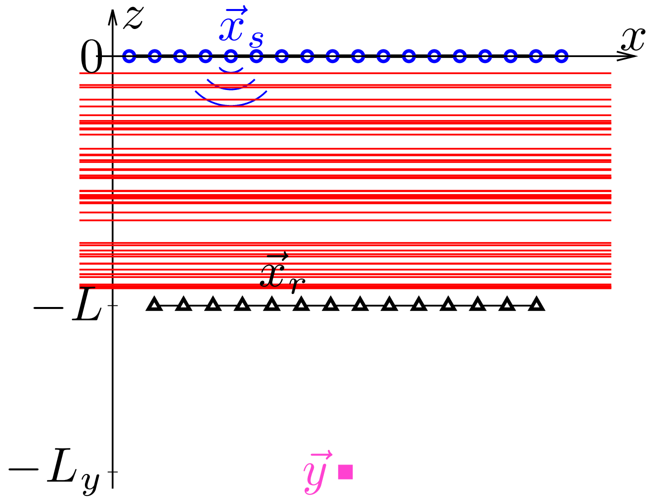
Consider the scaled regime "$l_c \ll \lambda \ll L$":

$$\omega \to \frac{\omega}{\varepsilon}, \qquad \mu(z) \to \mu\Big(\frac{z}{\varepsilon^2}\Big)$$

The moments of the random Green's function are known in the limit $\varepsilon \to 0$ [1].

[1] J.-P. Fouque, J. Garnier, G. Papanicolaou, and K. Sølna, *Wave propagation ...*, Springer, 2007.

Imaging below an overburden: analysis in the layered regime



- Assume that:

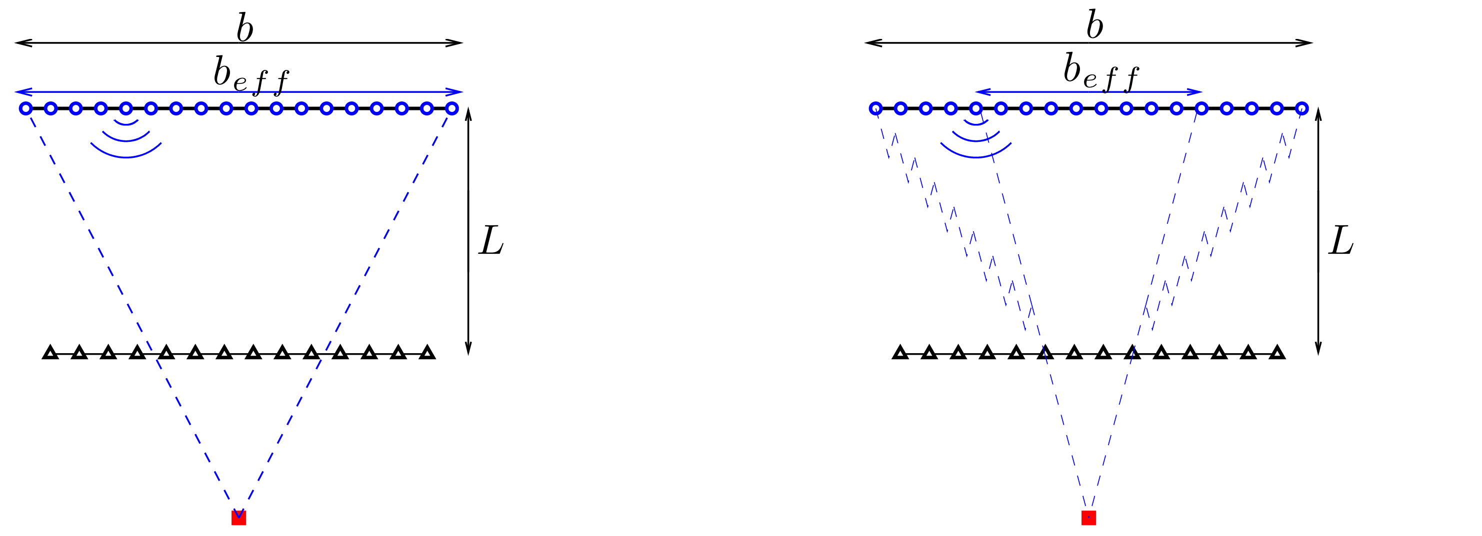
- the source aperture is b and the receiver aperture is a.

- there is a point reflector at $\vec{\boldsymbol{y}} = (\boldsymbol{y}, -L_y)$.

- the localization length $L_{\rm loc}$ is smaller than L (strong scattering, mean wave is damped):

$$L_{\rm loc} = \frac{4c_0^2}{\gamma\omega_0^2}, \qquad \gamma = \int_{-\infty}^{\infty} \mathbb{E}[\mu(0)\mu(z)]dz$$

Imaging below an overburden: analysis in the layered regime



Homogeneous medium

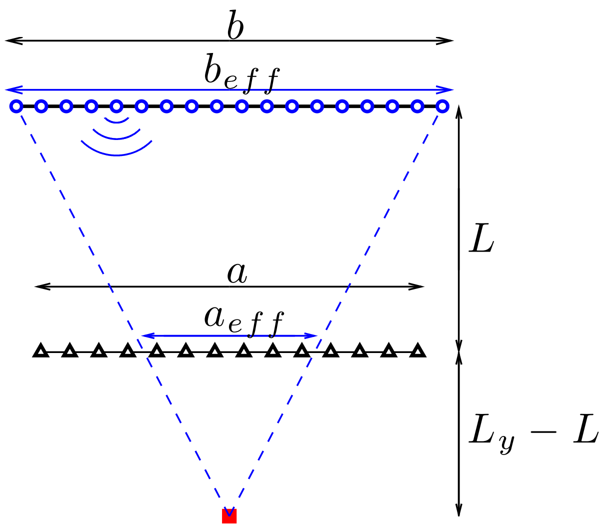
Randomly layered medium

Effective source aperture:

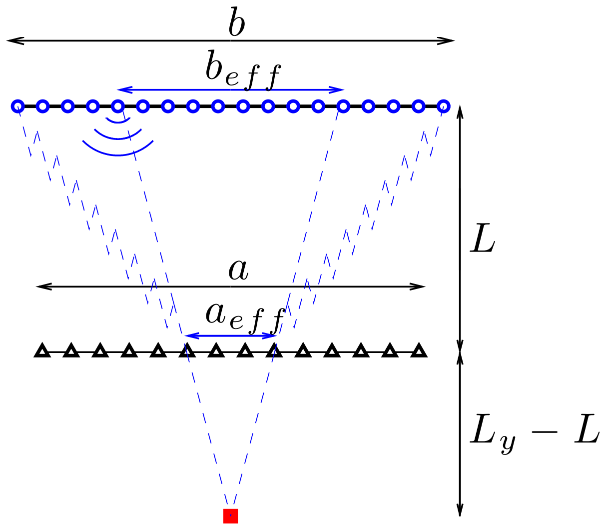
$b_{\text{eff}} = b$

$b_{\text{eff}}^2 = 4 L_{\text{loc}} L \ \ (\ll b^2)$

Imaging below an overburden: analysis in the layered regime



Homogeneous medium



Randomly layered medium

Effective source aperture:

$$b_{\text{eff}} = b \qquad\qquad b_{\text{eff}}^2 = 4 L_{\text{loc}} L \ \ (\ll b^2)$$

Effective receiver aperture:

$$a_{\text{eff}} = b \frac{L_y - L}{L_y} \qquad\qquad a_{\text{eff}} = b_{\text{eff}} \frac{L_y - L}{L_y}$$

Imaging below an overburden: analysis in the layered regime

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- The imaging function is statistically stable ($\lambda_0 \ll b, L$).

- The lateral resolution is $\dfrac{\lambda_0(L_y - L)}{a_{\rm eff}}$. The range resolution is $\dfrac{c_0}{B}\big(1 + \dfrac{B^2 L}{4\omega_0^2 L_{\rm loc}}\big)^{1/2}$.

- Since $a_{\rm eff}\mid_{\rm rand} < a_{\rm eff}\mid_{\rm homo}$, this shows that scattering does not help.

- physical reason: scattering reduces the angular and frequency diversity of the illumination.

Further results

• Use of other imaging functions based on cross-correlations (or Wigner distribution functions).

• Use of ambient noise sources.
One can apply correlation-based imaging techniques to signals emitted by ambient noise sources (increasingly popular in geophysics, "seismic interferometry").
↪ Travel time tomography (surface wave tomography since 2005, body waves more recently).
↪ Volcano monitoring (early warning of the eruption of Le Piton de la Fournaise in october 2010).
↪ Passive reflector imaging.

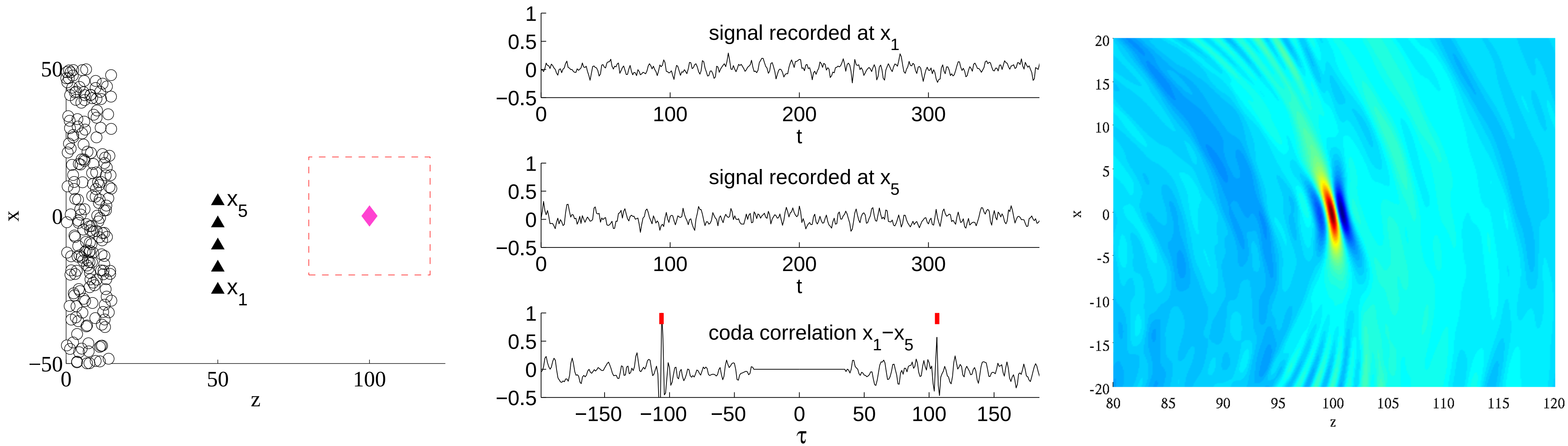
• Use of higher-order correlations.
One can apply imaging techniques based on special fourth-order cross correlations.
Useful when the statistics of the wave fluctuations is not Gaussian.

Passive sensor imaging of a reflector

- Ambient noise sources ($\circ$) emit stationary random signals.
- The signals $(u(t,\vec{\boldsymbol{x}}_r))_{r=1,\dots,N_{\rm r}}$ are recorded by the receivers $(\vec{\boldsymbol{x}}_r)_{r=1,\dots,N_{\rm r}}$ ($\blacktriangle$).
- The cross correlation matrix is computed and migrated:

$$\mathcal{I}(\vec{\boldsymbol{y}}^S) = \sum_{r,r'=1}^{N_{\rm r}} \mathcal{C}_T\big(\mathcal{T}(\vec{\boldsymbol{x}}_{r'},\vec{\boldsymbol{y}}^S)+\mathcal{T}(\vec{\boldsymbol{x}}_r,\vec{\boldsymbol{y}}^S),\vec{\boldsymbol{x}}_r,\vec{\boldsymbol{x}}_{r'}\big)$$

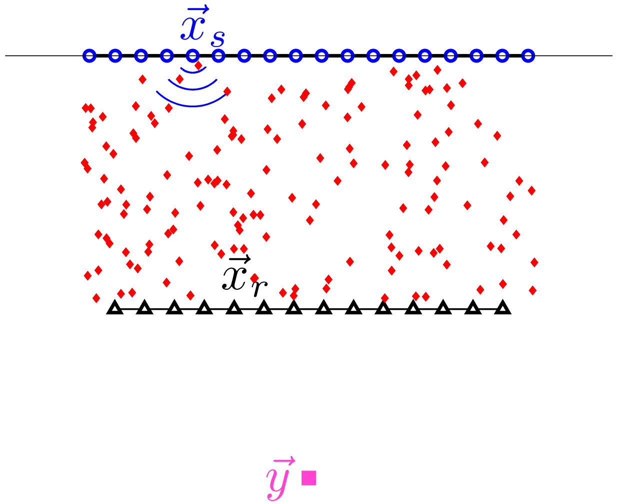
$$\text{with}\quad \mathcal{C}_T(\tau,\vec{\boldsymbol{x}}_r,\vec{\boldsymbol{x}}_{r'}) = \frac{1}{T}\int_0^T u(t+\tau,\vec{\boldsymbol{x}}_{r'})u(t,\vec{\boldsymbol{x}}_r)dt$$



Provided the ambient noise illumination is long (in time) and diversified (in angle and frequency): good stability [1].

[1] J. Garnier and G. Papanicolaou, *SIAM J. Imaging Sciences* **2**, 396 (2009).

Conclusions



• In scattering media one should migrate *well chosen* cross correlations of data, not data themselves.

• Method can be applied with ambient noise sources instead of controlled sources.

• Scattering can help ! Already noticed for time-reversal experiments, but far from clear in imaging problems.

Perspectives

- Space surveillance and imaging with airborne passive synthetic aperture arrays.